MATH 120A Prep: Proofs

1. For the following statements decide which proof technique is best to use to prove the statement and then write down the assumptions and conclusions you would need for the proof. You do not have to write the entire proof.

   (a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. Show if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

   Solution: We are assuming a negative statement and trying to prove a negative as well so we want to use contrapositive.
   Assumption: $f(x_1) = f(x_2)$. Conclusion: $x_1 = x_2$.

   (b) Prove there are infinitely many primes.

   Solution: To prove there are infinitely many we really want to show that there are not finitely many prime numbers. Thus to prove that this is not true we need to use proof by contradiction.
   Assumption: There are finitely many primes. Conclusion: Contradiction.

2. For the following statements decide which proof technique is best to use to prove the statement, then write out the proof.

   (a) Prove for all positive integers $n \geq 4$ that $3^n \geq n^3$.

   Solution: Since we are proving a statement for a set of positive integers we want to use induction.
   Base Case - $n = 4$: $3^4 = 81 \geq 64 = 4^3$. Induction: Suppose this is true for $n = k$. Then because $k \geq 4$,

   $$3^{k+1} = 3 \cdot 3^k \geq 3k^3 = k^3 + 2k^3 \geq k^3 + 3k^2 + 3k + 1 = (k+1)^3$$

   Therefore by induction $3^n \geq n^3$ for all $n \geq 4$.

   (b) Prove that $1 + \sqrt{2}$ is a root of the polynomial $x^2 - 2x - 1$.

   Solution: Being being a root is a positive statement that can be verified itself we can use direct proof by plugging $1 + \sqrt{2}$ into the polynomial:

   $$\begin{aligned}(1+\sqrt{2})^2 - 2(1+\sqrt{2}) - 1 &= (1 + 2\sqrt{2} + 2) - (2 + 2\sqrt{2}) - 1 \\ &= 0\end{aligned}$$

   so by direct proof $1 + \sqrt{2}$ is a root of $x^2 - 2x - 1$.